

Teaching and learning differential equations in engineering studies by using *low-level CAS commands*

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ABSTRACT: Differential equations are one of the major subjects taught to mathematics students in the second or third year of their undergraduate degrees in Kazakhstan. In addition to its importance to mathematics, differential equations are also required for physics, engineering, computer science and related disciplines. However, the current teaching of differential equations focuses on the algebraic solution rather than highlighting the importance of their applications. Using a computer algebra system (CAS) allows students to enhance their understanding of differential equations concepts by using *low-level CAS commands* to explain the solution processes of a differential equation. In the solutions of differential equations, the most important specific training is to develop the ability to place a differential equation in the proper class. An added advantage of CAS in the solutions of differential equations in general, and a Maple CAS in particular, is that students can classify equations by using the *odeadvisor* command in the Maple CAS. The authors discuss the use of the Maple CAS as a tool helping the student to achieve a better understanding of the differential equations concept.

INTRODUCTION

Differential Equations

The subject of differential equations (DE) constitutes a large and very important branch of modern mathematics. From the early days of calculus, the subject has been an area of great theoretical research and practical applications, and it continues to be so these days. In applications of mathematics to the technical sciences, differential equations occupy a particularly important place. They make it possible to solve many issues of general technical and special applied disciplines: physics, theoretical mechanics, material resistance, hydraulics, machine and mechanism theory, chemistry, production technology, biology, financial and economic, as well as other disciplines, and often emerge during problem solving. Numerous and diverse technical applications of the theory of ordinary differential equations require primarily knowledge of various physical and mechanical laws. The teaching and learning differential equations have been investigated for the last decade [1-5].

Artigue mentions that since the 17th Century, the theory of differential equations developed mathematically in several sets, such as:

1. Algebraic tools, where solutions are expressed through exact algebraic formulas, explicit or implicit, series expansions, integral expressions.
2. Numerical tools, where solutions are expressed through values of numerical approximations.
3. Geometric tools to characterise, from a topological point of view, the set of solution curves, that is, through the portrait of the equation - such a resolution is often called a qualitative solution [6].

Stephan and Rasmussen present an analysis of mathematical practices in the classroom, which they established during the instruction of first-order differential equations. The predictions are exponential in their form - as the authors call them - since they correspond to population growth models, which in this case is working with rabbits, and the predictions are based on an equation that is born from understanding the rate of change that models the scenario of the real world [7].

Arslan tells that *...since traditional DEs courses focus mainly on algebraic solutions and the techniques of algebraic solutions, and dismiss the conceptual understanding of DEs, students are satisfied with getting better in applying the algebraic techniques alone* [1].

Czocher presents a causal-comparative study of mixed methods of two instructional approaches in a course of differential equations for engineering students. He mentions *...that by having a better understanding of the role of conditions on the real world and the behavior of the phenomenon, it helps to make the appropriate conjectures in the analytical method*

of *undetermined coefficients* [8]. This type of modelling relies on the understanding of analytical methods; however, one needs to take into account that working with conditions in the real world helps to understand notions, such as the uniqueness of the solution. That is, in a problem modelled from Newton's law of cooling, hypothesis-generating questions can arise when the uniqueness of the solution is conflicted in considering the real world.

Maple Computer Algebra System

Maple is a mathematical computer program, designed to perform a variety of mathematical calculations and operations on symbolic, as well as numeric entities. There are several examples of using CAS in higher education, including specific developments in teaching. One of the promising educational areas in which the use of information technologies seems to be the most promising is mathematics [9][10].

Physics teachers also began to pay more attention to the use of CAS in theoretical courses in general, and differential equations in particular, since when solving most physical problems, especially in theoretical physics courses, it is necessary to perform bulky mathematical calculations, which can cause some difficulties for students [11][12].

How does Maple CAS help to teach and learn differential equations?

The Maple CAS is an indispensable *assistant* in teaching various sections of mathematics. This applies to the subject *Ordinary Differential Equations*. Most of the intermediate steps of solving a problem used by the *dsolve* command are available to the user, including all steps related to the use of symmetry groups and an integrating factor. Additional parameters in the *dsolve* command allow ordinary differential equations to be solved using methods selected by the user, with or without an integration step. All this makes it easier to understand both the possible different methods of solving the problem and the application of the *dsolve* command and the associated Maple commands, and can be successfully used to work in a computer class.

Maple has a special DEtools package, which is mainly designed for professional work with ordinary differential equations. Using the functions of this package, one can transform differential equations using integrating factors and Lie symmetry groups, lower the order of the equation in question, calculate singular points, convert high-order equations to a system of first-order equations, classify ordinary differential equations, and much more. A large group of functions in this package is designed to visualise numerical solutions to the Cauchy problem for ordinary differential equations and systems of such equations.

The functions of the DEtools package can be used for educational purposes. The authors will briefly focus on these features of the package now. Packages are loaded by entering the command:

```
with(packagename).
```

To access DEtools features, one must enter:

```
> with(DEtools):
```

or load the desired function, for example, the *odeadvisor* command, using the following:

```
> with(DEtools, odeadvisor):
```

In the solutions of differential equations, the most important specific training is to develop the ability to place a differential equation in the proper class. The *odeadvisor* command plays an important role in training. The main purpose of this function is to classify this equation according to the standards outlined in the reference books for ordinary differential equations and to display a reference page containing information for solving the equation. Reference pages include examples and a Maple input string, along with some advice that allows the user to adapt this information to their task.

Conceptual Understanding

The development and improvement of a conceptual understanding of mathematical content are one of the important goals of teaching differential equations. Various studies have confirmed the students' difficulties in understanding the concepts of differential equations [13-15].

The mathematical activity of students, including operational and algorithmic actions and forming skills tasks, involve: formulating algorithms, performing actions on ready-made algorithms, checking and correcting algorithms, combining algorithms, breaking complex tasks into subtasks, and developing ideas about algorithmic processes and methods for their description.

The potential of the subject of differential equations in the context of the formation of algorithmic competence of future engineers lies in the use of CAS in differential equations classes, contributing to the transition of teaching differential equations to a qualitatively new level (reducing the time spent on mathematical calculations, effective verification of

the correctness of the solution, choosing the optimal solution, approximate calculations, the possibility of plotting complex functions, etc) [16].

METHODOLOGY

This part of the present study addresses whether and how students can develop conceptual understanding through problem solving using CAS *low-level* commands rather than procedure-based learning using predefined algorithms in the DE subject. Maple can solve a variety of non-linear first-order equations that are typically encountered in the introductory differential equation subject. The use the *odeadvisor* function, which is contained in the DEtools package, is to help students classify equations. No output means that Maple cannot solve the problem, that the problem has no solution. Generally, when Maple returns nothing, one should try other methods to determine if the problem has solutions that Maple cannot find.

ELECTRIC CIRCUIT PROBLEMS

In this section, the authors consider the application of differential equations to a series circuit containing: 1) an electromotive force; and 2) a resistor, inductor and capacitor [3][17]. Let simply recall that the electromotive force (for example, a battery or generator) produces a flow of current in a closed circuit and that this current produces the so-called voltage drop across each element: a resistor, inductor and capacitor. Further, the following three laws concerning the voltage drops across these various elements are known to hold:

1. The voltage drop across a resistor is given by $E_R = Ri$, where R is a constant of proportionality called the resistance, and i is the current.
2. The voltage drop across an inductor is given by $E_L = L \frac{di}{dt}$, where L is a constant of proportionality called the inductance, and i again denotes the current.
3. The voltage drop across a capacitor is given by $E_C = \frac{1}{C}q$, where C is a constant of proportionality called the capacitance and q is the instantaneous charge in the capacitor. Since $i = \frac{dq}{dt}$ this is often written as $E_C = \frac{1}{C} \int idt$.

A circuit has in series an electromotive force given by $E = 100 \sin 40t$ V, a resistor of 10Ω , and an inductor of 0.5 H. If the initial current is 0, find the current at time $t > 0$.

Formulation: i denotes the current in amperes at time t . The total electromotive force is $100 \sin 40t$ V. Applying Kirchhoff's voltage law, one finds that the voltage drops are as follows:

1. Across the resistor: $E_R = Ri = 10i$.
2. Across the inductor: $E_L = L \frac{di}{dt} = \frac{1}{2} \frac{di}{dt}$

Applying Kirchhoff's law, have the differential equation:

$$\begin{aligned} \frac{1}{2} \frac{di}{dt} + 10i &= 100 \sin 40t, \text{ or} \\ \frac{di}{dt} + 20i &= 200 \sin 40t. \end{aligned} \tag{1}$$

Since the initial current is 0, the initial condition is:

$$i(0) = 0. \tag{2}$$

Solution: Equation (1) is a first-order linear equation. An integrating factor is:

$$e^{\int 20dt} = e^{20t}.$$

Multiplying equation (1) by this, one obtains:

$$\begin{aligned} e^{20t} \frac{di}{dt} + 20e^{20t}i &= 200e^{20t} \sin 40t \\ \frac{d}{dt}(e^{20t}i) &= 200e^{20t} \sin 40t \end{aligned}$$

Integrating and simplifying, one can find:

$$i = 2(\sin 40t - 2 \cos 40t) + Ce^{-20t}.$$

Applying the condition (2), $i = 0$ when $t = 0$, one can find that $C = 4$. Thus, the solution is:

$$i = 2(\sin 40t - 2 \cos 40t) + 4e^{-20t}.$$

Expressing the trigonometric terms in a *phase-angle* form, one can have:

$$i = 2\sqrt{5} \left(\frac{1}{\sqrt{5}} \sin 40t - \frac{2}{\sqrt{5}} \cos 40t \right) + 4e^{-20t}$$

$$\text{or } i = 2\sqrt{5} \sin(40t + \phi) + 4e^{-20t}, \quad (3)$$

where ϕ is determined by the equation:

$$\phi = \arccos \frac{1}{\sqrt{5}} = \arcsin \left(-\frac{2}{\sqrt{5}} \right).$$

One finds $\phi \approx -1.11$ rad, and hence, the current is given approximately by:

$$i = 4.47(\sin 40t - 1.11) + 4e^{-20t}.$$

Interpretation: the current is clearly expressed as the sum of a sinusoidal term and an exponential one. The exponential one becomes so very small in a short time that its effect is soon practically negligible; it is a transient term.

Thus, after a short time essentially all that remains is the sinusoidal term; it is the steady-state current. Observe that its period $\frac{\pi}{20}$ is the same as that of the electromotive force. However, the phase angle $\phi \approx -1.11$ indicates that the

electromotive force leads the steady-state current by approximately $\frac{1}{40}(1.11)$.

Here, the authors include examples of using Maple to see and check solutions step by step.

```

> eq:=-diff(i(t),t)+20*i(t)=200*sin(40*t);
> init:=i(0)=0;
> dsolve({eq,init},i(t));
> with(DEtools);
> odeadvisor(eq);
> dsolve(eq,[linear],useInt);
> simplify(value(%));
> mu:=intfactor(eq);
> mu*eq;
> dsolve(mu*eq,[exact]);
> simplify(%);

```

$$eq = \left(\frac{d}{dt} i(t) \right) + 20 i(t) = 200 \sin(40 t)$$

$$init = i(0) = 0$$

$$i(t) = 2 \sin(40 t) - 4 \cos(40 t) + 4 e^{-20 t}$$

$$[[_linear, class A]]$$

$$i(t) = e^{-20 t} \int 200 \sin(40 t) e^{20 t} dt + e^{-20 t} _C1$$

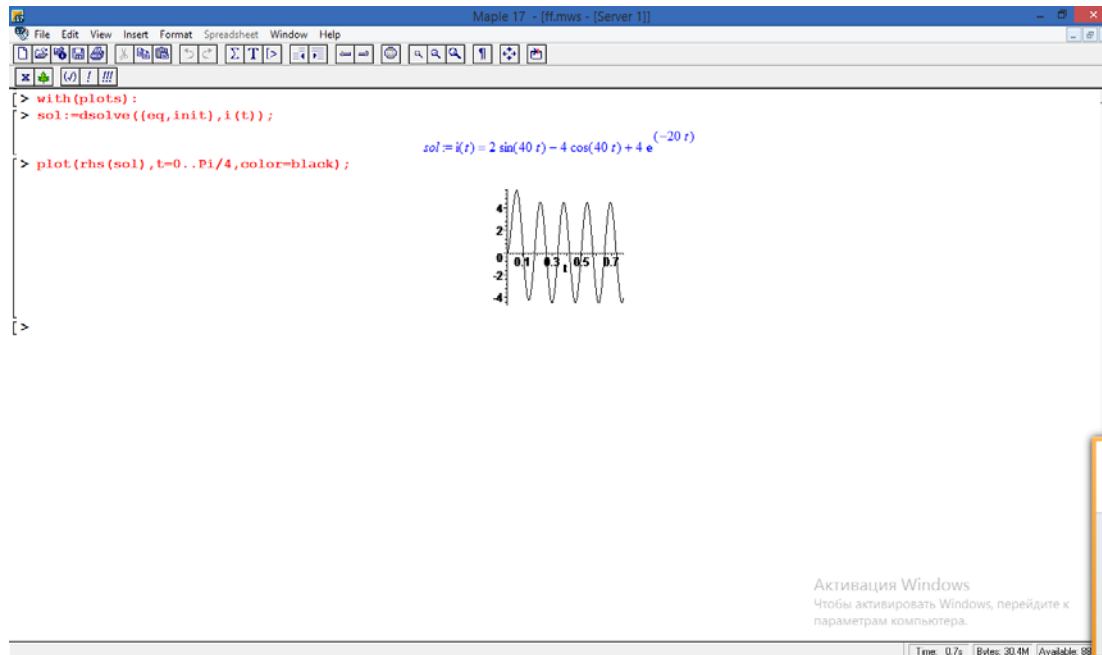
$$i(t) = e^{-20 t} (2 \sin(40 t) e^{20 t} - 4 \cos(40 t) e^{20 t} + _C1)$$

$$\mu = e^{20 t}$$

$$e^{20 t} \left(\left(\frac{d}{dt} i(t) \right) + 20 i(t) \right) = 200 \sin(40 t) e^{20 t}$$

$$i(t) = \frac{2 \sin(40 t) e^{20 t} - 4 \cos(40 t) e^{20 t} - _C1}{e^{20 t}}$$

$$i(t) = 2 \sin(40 t) e^{-20 t} - 4 \cos(40 t) e^{-20 t} - _C1 e^{-20 t}$$



> eq:=diff(i(t),t)+20*i(t)=200*sin(40*t);

$$eq := \left(\frac{d}{dt} i(t) \right) + 20 i(t) = 200 \sin(40 t)$$

> init:=i(0)=0;

$$init := i(0) = 0$$

> dsolve({eq,init},i(t));

$$i(t) = 2 \sin(40 t) - 4 \cos(40 t) + 4 e^{(-20 t)}$$

> with(DEtools):
> odeadvisor(eq);

$$[[_linear, class A]]$$

> dsolve(eq,[linear],useInt);

$$i(t) = e^{(-20 t)} \int 200 \sin(40 t) e^{(20 t)} dt + e^{(-20 t)} _C1$$

> simplify(value(%));

$$i(t) = e^{(-20 t)} (2 \sin(40 t) e^{(20 t)} - 4 \cos(40 t) e^{(20 t)} + _C1)$$

> mu:=intfactor(eq);

$$\mu := e^{(20 t)}$$

> mu*eq;

$$e^{(20 t)} \left(\left(\frac{d}{dt} i(t) \right) + 20 i(t) \right) = 200 \sin(40 t) e^{(20 t)}$$

> dsolve(mu*eq,[exact]);

$$i(t) = \frac{2 \sin(40 t) e^{(20 t)} - 4 \cos(40 t) e^{(20 t)} - _C1}{e^{(20 t)}}$$

> simplify(%);

$$i(t) = (2 \sin(40 t) e^{(20 t)} - 4 \cos(40 t) e^{(20 t)} - _CI) e^{(-20 t)}$$

```
> with(plots):
> sol:=dsolve({eq,init},i(t));

sol := i(t) = 2 sin(40 t) - 4 cos(40 t) + 4 e(-20 t)

> plot(rhs(sol),t=0..Pi/4,color=black);
```

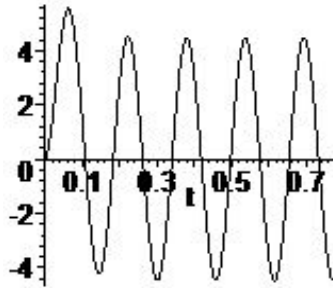


Figure 1: Sketching graphics with Maple.

CONCLUSIONS

Maple is one of computer algebra systems and is now widely used in engineering practice, in education and research. In this article, the authors discuss the advantages of using *low-level* commands of Maple CAS as a tool for a better understanding of application problems of differential equations. Teaching ideas shared in this article may be of an interest to teachers who want to explore differential equations in engineering education.

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